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2D and 3D Loci Inspired by an Entrance Problem and Technologies

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Abstract

In this paper, we discuss a locus problem that was originated from Chinese college entrance exam practice problems [5] and it has been discussed in [4]. We will see how the 2D locus problem can be explored using the dynamic geometry software (DGS) Geometry in Mathematical Arts [2] with different strategies. Next, we extend the 2D locus problem to more challenging corresponding problems in 3D with the help of a DGS [2]. We also illustrate how a computer algebra system (CAS) Maple [3] can be used to derive our locus analytically. We shall see that the use of a DGS in constructing the locus is very accessible to students when they can visualize what the locus might look like first . On the other hand, when readers need to use a CAS for verifying if results are consistent with our visualization, the task becomes much more challenging. In particular, the process of finding three points on the ellipsoid systematically and constructing a set of three linearly independent vectors requires the knowledge of a rotation matrix, whose computation is tedious if a CAS is not available. Once the rotation matrix is known, it is then simple to visualize the rotation of a vector about an axis, an important concept in computer graphics. The paper shows that with appropriate aids of technological tools, challenging and applicable mathematics can be made more fun and accessible.

1 Introduction

In *[4]*, we considered the problem that appeared at a practice problem of a College Entrance Exam mentioned in *[5]*, which we state as follows: We are given a circle of radius 1 centred at the origin, and choose a point $C = (a, 0)$ with $0 < a < 1$. Let D and E to be two points on the circle such that the angle $\measuredangle DCE$ is a right angle. Let G be chosen so that $DCEG$ is a rectangle. The question is to ask for the locus of G. We remark that the original problem stated in *[5]* is to find the point G when the radius $r = 6$ and $a = 4$. In section 2, we use GInMA *[2]* to explore a variation of this two-dimensional scenario and use the CAS Maple *[3]* to verify the locus analytically. In section 3, we further extend the planar problem to a more challenging one in space that is accessible for university students who have learned the concepts of multivariable calculus and linear algebra. Specifically, we replace the 2D ellipse case to a corresponding 3D ellipsoid. We remark that the techniques adopted in finding the locus surface in 3D in this paper certainly can be extended to even higher dimensions and we leave this to readers to do further investigations.

2 The 2D scenario

In this section, we will replace the original unit circle by an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, shown in blue in Figure 1, we further assume the ratio of semi axes $b = \frac{OB}{OA} < 1$. The point $C = (c, 0)$ is fixed on the longer semiaxis OA with $c = \frac{OC}{OA} < 1$. Next we pick two points D and E on the ellipse (OAB). We are given a fixed angle $\beta = \measuredangle DCE$ (see Figure 1), which is not necessary a right angle from the original scenario. If we let $\varphi = \measuredangle DCA$. We shall investigate the locus of $\overrightarrow{CG} = \overrightarrow{CD} + \overrightarrow{CE}$ while all points D corresponding to the angle $\varphi \in [0, 2\pi]$.

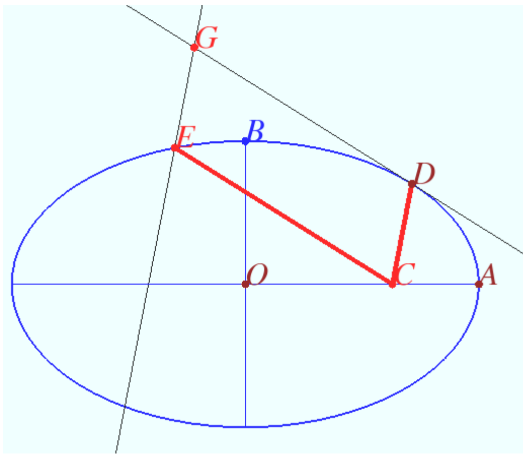


Figure 1. Original ellipse and the point G.

First, we consider the vector $\overrightarrow{CD}$ and let $\left\|\overrightarrow{CD}\right\| = k$, then $D = (c + k\cos\varphi, k\sin\varphi)$. Since D is a point on the ellipse, we see

$$\frac{(c + k\cos\varphi)^2}{a^2} + \frac{(k\sin\varphi)^2}{b^2} = 1. \tag{1}$$

We solve for k and choose the positive root from Eq. (1). After simplification, we find k

$$k = \frac{-cb^2\cos\varphi + b\sqrt{(b^2 + c^2 - a^2)\cos^2\varphi + (a^2 - c^2)}}{a^2 - (a^2 - b^2)\cos^2\varphi}. \tag{2}$$

Similarly, if we consider the vector $\overrightarrow{CE}$ and let $\left\|\overrightarrow{CE}\right\| = l$, then $E = (c + l\cos(\varphi + \beta), l\sin(\varphi + \beta))$. Since E is a point on the ellipse, we have

$$\frac{(c + l\cos(\varphi + \beta))^2}{a^2} + \frac{(l\sin(\varphi + \beta))^2}{b^2} = 1. \tag{3}$$

We solve for l and choose the positive root from Eq. (3). After simplification, we discover

$$l = \frac{-cb^2 \cos(\varphi+\beta) + b\sqrt{(b^2+c^2-a^2)\cos^2(\varphi+\beta) + (a^2-c^2)}}{a^2 - (a^2-b^2)\cos^2(\varphi+\beta)}. \tag{4}$$

Finally, we find the locus G by using $\overrightarrow{CG} = \overrightarrow{CD} + \overrightarrow{CE}$.

Example 1 *We are given a fixed ellipse in blue* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$*, and we assume the ratio of semi axes* $b = \frac{OB}{OA} = 0.6$*. The point* $C = (0.5, 0)$ *is fixed on the semiaxis* OA*. Next we pick two points* D *and* E *on the ellipse* (OAB) *such that* $\beta = \measuredangle DCE = \frac{\pi}{3}$*. The locus of* $\overrightarrow{CD} + \overrightarrow{CE}$ *using Maple [3] and GInMA [2] are shown in Figure 2(a) and Figure 2(b), respectively. We leave readers to explore the complete analytic solution including animation using Maple [3] in [S1], and the GInMA file using [2] in [S2].*

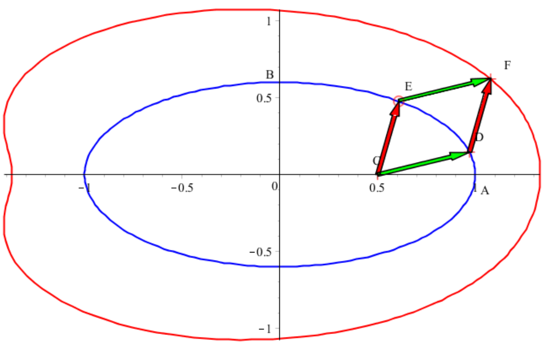


Figure 2(a). Maple and the locus for ellipse.

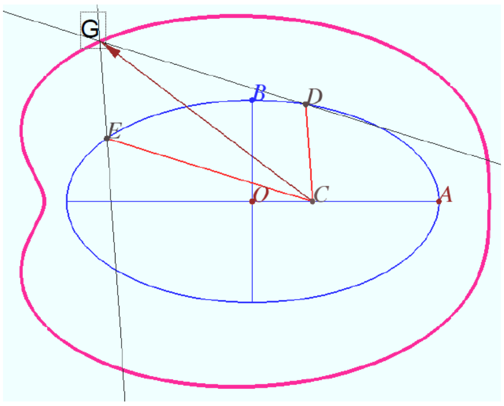


Figure 2(b). GInMA and the locus for ellipse.

3 The 3D Locus for an ellipsoid

We consider an ellipsoid $OABC$ of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with $OA = a = 1, 0 < OB = b \leq 1, 0 < OC = c \leq 1)$ and the fixed point $D = (d, 0, 0)$ is given, where $0 < d \leq 1$. We are given E to be a moving point on the ellipsoid. To generalize the planar scenario to three-dimensions, we need to know how to select a point E on the ellipsoid to begin with and how to pick G and F systematically with the help of the vector $\overrightarrow{DE}$ so that the points E, G and F are all on the ellipsoid and $\left\{\overrightarrow{DE}, \overrightarrow{DF}, \overrightarrow{DG}\right\}$ forms a linearly independent set. Our final objective is to find the locus of $\overrightarrow{DE} + \overrightarrow{DF} + \overrightarrow{DG}$.

3.1 Construction of the 3D locus with a CAS

The challenging task to extend the 2D scenario to the corresponding 3D case is to see how the desired points E, F and G can be chosen on the given ellipsoid. We further remark that we do not randomly construct a linearly independent set $\left\{\overrightarrow{DE}, \overrightarrow{DF}, \overrightarrow{DG}\right\}$. Instead, we will describe how we choose the point E on the ellipsoid from the given fixed point D, and show how we use appropriate rotating axes to construct subsequent points of G and F, respectively. First, we describe our coordinate system and appropriate angles. Let $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ represent the x, y and z axes respectively. We choose E to be a point on the ellipsoid $OABC$ and E' be the projection of $\overrightarrow{OE}$ onto the $yz-$plane. For notation purpose, we refer to Figure 3 and let the angle $\varphi = \measuredangle EOE'$. The γ is the angle between $\overrightarrow{OE'}$ and the positive $y - axis$ (in other words, $\gamma = \measuredangle E'OB$ in Figure 3). Thus we may write the parametric equation of the ellipsoid $OABC$ as $(a \sin\varphi, b\cos\varphi\cos\gamma, c\sin\varphi\cos\gamma)$, where $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\gamma \in [0, 2\pi]$.

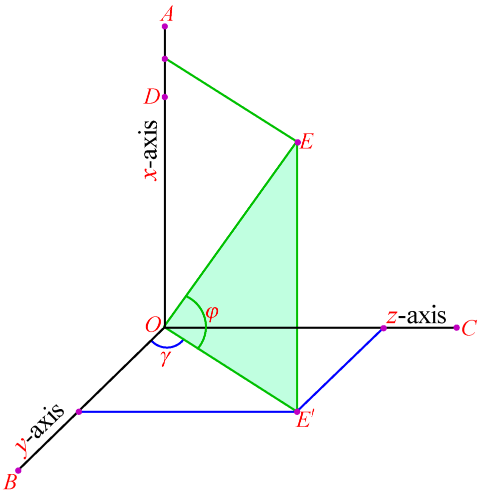


Figure 3. Spherical coordinates OA, OB and OC with $\varphi = \measuredangle EOE'$ and $\gamma = \measuredangle E'OB$.

We describe how we find the locus of $\overrightarrow{DE} + \overrightarrow{DF} + \overrightarrow{DG}$ analytically in the following steps. Our general strategy is to construct an orthogonal set of $\left\{\overrightarrow{DE}, \overrightarrow{v_1}, \overrightarrow{v_2}\right\}$. Next we rotate $\overrightarrow{DE}$ by a non-zero angle α about the axis $\overrightarrow{v_1}$ to obtain the point F on the ellipsoid. Similarly, we rotate $\overrightarrow{DE}$ by a nonzero angle β about the axis $\overrightarrow{v_2}$ to reach the point G on the ellipsoid. When $\overrightarrow{DE}, \overrightarrow{DF}$ and $\overrightarrow{DG}$ are not coplanar, then we see that $\left\{\overrightarrow{DE}, \overrightarrow{DF}, \overrightarrow{DG}\right\}$ forms a linearly independent set for any $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\gamma \in [0, 2\pi]$.

1. We construct the point E on the ellipsoid then such that

$$\overrightarrow{OE} = (a\sin\varphi, b\cos\varphi\cos\gamma, c\sin\varphi\cos\gamma)$$

and

$$\overrightarrow{DE} = (a\sin\varphi - d, b\cos\varphi\cos\gamma, c\sin\varphi\cos\gamma)$$

(see Figure 4).

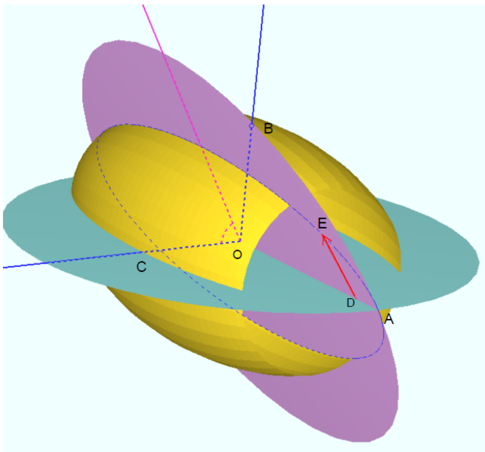


Figure 4. The point E, planes OAC and OAE using GInMA.

2. We define $\overrightarrow{v_1}$ to be the unit normal vector of the plane OAE that is based at the point D. In other words, we write $\overrightarrow{v_1} = \frac{\overrightarrow{OD}\times\overrightarrow{DE}}{\|\overrightarrow{OD}\times\overrightarrow{DE}\|} = (v_{11}, v_{12}, v_{13})$.

3. We define $\overrightarrow{v_2}$ to be the unit vector starting from the point D, satisfying $\overrightarrow{v_2} \perp \overrightarrow{v_1}$ and $\overrightarrow{v_2} \perp \overrightarrow{DE}$. In other words, we may write $\overrightarrow{v_2} = \frac{\overrightarrow{DE}\times\overrightarrow{v_1}}{\|\overrightarrow{DE}\times\overrightarrow{v_1}\|} = (v_{21}, v_{22}, v_{23})$. Recall that (see *[1]* or *[6]*) a rotation matrix with respect to a rotation axis can be derived as follows: In $\mathbb{R}^2$ we know that the rotation matrix of a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ at an angle θ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Now in $\mathbb{R}^3$, we first start with a unit vector $\overrightarrow{U_2}$, and use the common strategy of constructing a three-dimensional orthonormal set $\{\overrightarrow{U_0}, \overrightarrow{U_1}, \overrightarrow{U_2}\}$. Suppose the unit vector $\overrightarrow{U_2} = (a, b, c)$, then we write
$$\overrightarrow{U_0} = \frac{(b, -a, 0)}{\sqrt{a^2 + b^2}}, \tag{5}$$
and let
$$\overrightarrow{U_1} = \overrightarrow{U_2} \times \overrightarrow{U_1} = \frac{(ac, bc, -a^2 - b^2)}{\sqrt{a^2 + b^2}}. \tag{6}$$
We see the set $\{\overrightarrow{U_0}, \overrightarrow{U_1}, \overrightarrow{U_2}\}$ is a right-handed orthonormal set and would like to find a rotation matrix R corresponding to a rotation by an angle θ about the axis $\overrightarrow{U_2}$. In other words, the vector $\overrightarrow{U_2}$ is invariant under the rotation matrix R and for $\overrightarrow{V} = x_0\overrightarrow{U_0} + x_1\overrightarrow{U_1} +$

$x_2\overrightarrow{U_2} \in \mathbb{R}^3$, where $x_i = \overrightarrow{U_i} \cdot \overrightarrow{V}$ with $i = 0, 1$ and 2. The rotation of $\overrightarrow{V}$ will be

$$\begin{aligned} R\overrightarrow{V} &= (x_0\cos\theta - x_1\sin\theta)\overrightarrow{U_0} + (x_0\sin\theta + x_1\cos\theta)\overrightarrow{U_1} + x_2\overrightarrow{U_2} && (7)\\ &= U\begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_0\\ x_1\\ x_2 \end{bmatrix} && (8)\\ &= U\begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1 \end{bmatrix}U^TV, && (9) \end{aligned}$$

where $U = [\overrightarrow{U_0} : \overrightarrow{U_1} : \overrightarrow{U_2}]$. Therefore, the rotation matrix for a rotation by an angle θ about an axis is

$$R = U\begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1 \end{bmatrix}U^T. \tag{10}$$

When the axis of rotation $\overrightarrow{U_2}$ is (a, b, c) and $\overrightarrow{U_0}$ and $\overrightarrow{U_1}$ are chosen as in Eqs. (5) and (6), respectively, the rotation matrix can be shown to be

$$R = \begin{bmatrix} a^2(1-\cos\theta)+\cos\theta & ab\,(1-\cos\theta)-c\sin\theta & ac\,(1-\cos\theta)+b\sin\theta\\ ab\,(1-\cos\theta)+c\sin\theta & b^2(1-\cos\theta)+\cos\theta & bc\,(1-\cos\theta)-a\sin\theta\\ ac\,(1-\cos\theta)-b\sin\theta & bc\,(1-\cos\theta)+a\sin\theta & c^2(1-\cos\theta)+\cos\theta \end{bmatrix} \tag{11}$$

4. We are ready to define the point F explicitly. The point F should be picked so that it lies on the ellipsoid and the plane OAE; in addition, $\overrightarrow{DF}$ is the vector obtained by rotating $\overrightarrow{DE}$ around the axis $\overrightarrow{v_1}$, which is the normal vector of the plane OAE, by a pre-determined angle α (see Figure 5). In other words, α is the angle between $\overrightarrow{DE}$ and $\overrightarrow{DF}$. In view of Eq. (11), we consider the rotation matrix

$$A = \begin{bmatrix} \cos\alpha+(1-\cos\alpha)\,v_{11}^2 & (1-\cos\alpha)\,v_{11}v_{12}-\sin\alpha v_{13} & (1-\cos\alpha)\,v_{11}v_{13}+\sin\alpha v_{12}\\ (1-\cos\alpha)\,v_{11}v_{12}+\sin\alpha v_{13} & \cos\alpha+(1-\cos\alpha)\,v_{12}^2 & (1-\cos\alpha)\,v_{12}v_{13}-\sin\alpha v_{11}\\ (1-\cos\alpha)\,v_{11}v_{13}-\sin\alpha v_{12} & (1-\cos\alpha)\,v_{12}v_{13}+\sin\alpha v_{11} & \cos\alpha+(1-\cos\alpha)\,v_{13}^2 \end{bmatrix} \tag{12}$$

and define the following vector

$$\overrightarrow{u} = \frac{A\left(\overrightarrow{DE}\right)}{\left\|A\left(\overrightarrow{DE}\right)\right\|} = (u_1, u_2, u_3)\,. \tag{13}$$

If we use the substitutions of

$$a_{11} = u_1^2 + \frac{u_2^2}{b^2} + \frac{u_3^2}{c^2}, b_{11} = \frac{du_1}{a_{11}}, l_1 = \sqrt{b_{11}^2 + \frac{(1-d^2)}{a_{11}}} - b_{11}, \tag{14}$$

and let

$$\overrightarrow{OF} = l_1\overrightarrow{u} + \overrightarrow{OD}. \tag{15}$$

Then F will be a point lying on the ellipsoid and on the plane of the plane OAE. Furthermore, $\overrightarrow{DF}$ is the vector obtained by rotating $\overrightarrow{DE}$ around the rotating axis $\overrightarrow{v_1}$ by an angle α.

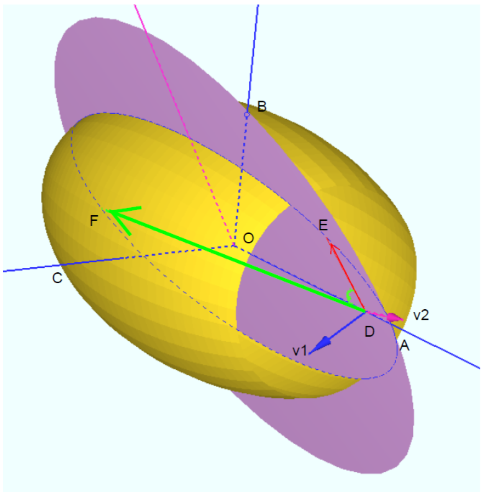


Figure 5. The point F, vectors v_1 and v_2.

5. We construct the point G on the ellipsoid in a similar way as we did for the point F. We need to choose the point G so that it also lies on the plane with normal vector $\overrightarrow{v_2}$, or is spanned by vectors $\overrightarrow{DE}$ and $\overrightarrow{v_1}$. To do this, we rotate $\overrightarrow{DE}$ around the rotating axis $\overrightarrow{v_2}$ with a pre-determined angle β. In other words, we have $\alpha = \measuredangle EDF$ and $\beta = \measuredangle EDG$ (see Figure 6).

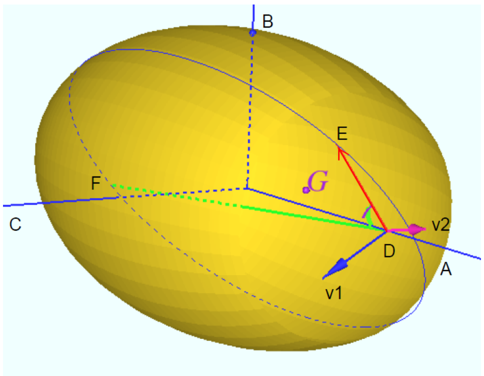


Figure 6. The point G, vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.

6. We describe how we obtain G analytically here. Analogous to our choice of matrix A in

Eq. (12), we consider the rotation matrix

$$B = \begin{bmatrix} \cos\beta + (1-\cos\beta)\, v_{21}^2 & (1-\cos\beta)\, v_{21}v_{22} - \sin\beta v_{23} & (1-\cos\beta)\, v_{21}v_{23} + \sin\beta v_{22} \\ (1-\cos\beta)\, v_{21}v_{22} + \sin\beta v_{23} & \cos\beta + (1-\cos\alpha)\, v_{22}^2 & (1-\cos\beta)\, v_{22}v_{23} - \sin\beta v_{21} \\ (1-\cos\beta)\, v_{21}v_{23} - \sin\beta v_{22} & (1-\cos\beta)\, v_{22}v_{23} + \sin\beta v_{21} & \cos\beta + (1-\cos\beta)\, v_{23}^2 \end{bmatrix} \tag{16}$$

and consider the rotated vector

$$\overrightarrow{w} = \frac{B\left(\overrightarrow{DE}\right)}{\left\| B\left(\overrightarrow{DE}\right)\right\|} = (w_1, w_2, w_3)\,. \tag{17}$$

If we let $a_{22} = w_1^2 + \frac{w_2^2}{b^2} + \frac{w_3^2}{c^2}, b_{22} = \frac{dw_1}{a_{22}}$ and $l_2 = \sqrt{b_{22}^2 + \frac{(1-d^2)}{a_{22}}} - b_{22}$, and set

$$\overrightarrow{OG} = l_2\overrightarrow{w} + \overrightarrow{DE}. \tag{18}$$

Then G will be the desired point on the ellipsoid satisfying conditions mentioned in item 5 above.

7. Finally, we define the locus H to be

$$\overrightarrow{OH} = \overrightarrow{OD} + \overrightarrow{DE} + \overrightarrow{u} + \overrightarrow{w}. \tag{19}$$

Example 2 *We consider the ellipsoid $OABC$ of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with $OA = a = 1, OB = b = 0.75, OC = c = 0.5$) and the given point $D = (d, 0, 0)$, where $d = 0.75$. We let the point E to be a moving point on the ellipsoid. We shall find the points G and F on the ellipsoid so that $\left\{\overrightarrow{DE}, \overrightarrow{DG}, \overrightarrow{DF}\right\}$ forms a linearly independent set. Then find the locus of $\overrightarrow{DE} + \overrightarrow{DG} + \overrightarrow{DF}$.*

We follow the steps described in Section 3.1 to find the points G and F on the ellipsoid so that $\left\{\overrightarrow{DE}, \overrightarrow{DG}, \overrightarrow{DF}\right\}$ forms a linearly independent. To start withe a point E, we let $\varphi = 0.3$ radian and $\gamma = \frac{\pi}{4}$. To find the point F and the point G, we choose $\alpha = \frac{\pi}{6}$ and $\beta = \frac{\pi}{4}$. The complete selection of the point E, calculations of the points F and G, and the locus surface of $\overrightarrow{DE} + \overrightarrow{DG} + \overrightarrow{DF}$ can be found from [S3]. To facilitate the computations in Maple, we introduce the substitution $r = \frac{bc}{\sqrt{b^2 + c^2\tan^2\gamma}}$. It is easy to check that if we write $E = [\sin\varphi, r\cos\varphi\tan\gamma, r\cos\varphi]$, then E is indeed on the ellipsoid of $(\sin\varphi, b\cos\varphi\cos\gamma, c\cos\varphi\sin\gamma)$, where $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\gamma \in [0, 2\pi]$.When E and other corresponding angles are chosen, we see the calculations of $E, \overrightarrow{v_1}$ and $\overrightarrow{v_2}$ in Figure 7(a) and the vectors $\overrightarrow{DE}, \overrightarrow{DG}, \overrightarrow{DF}$ together with the original ellipsoid is shown in Figure 7(b) below.

"b=", 0.75, ", c=", 0.5, ", d =", 0.75, ", alpha=", 30, ", beta=", 45, ", gamma=", 45, "phi0 = ", 0.3

$$E := \begin{bmatrix} 0.2955202067 \\ 0.3974440035 \\ 0.3974440035 \end{bmatrix}$$

$$v1 := \begin{bmatrix} 0. \\ -0.7071067810 \\ 0.7071067810 \end{bmatrix}$$

$$v2 := \begin{bmatrix} 0.777603205607363 \\ 0.444597152104034 \\ 0.444597152104034 \end{bmatrix}$$

Figure 7(a). The point E and vectors v_1 and v_2.

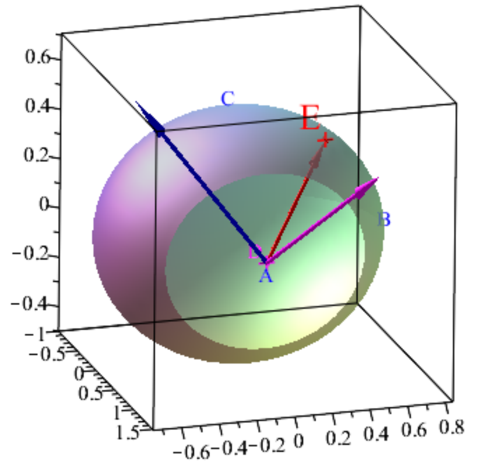


Figure 7(b). The vectors of $\overrightarrow{DE}, \overrightarrow{DG}$ and $\overrightarrow{DF}$.

Furthermore, the vectors u, w, the points F, G and H and the vector of $\overrightarrow{DH} = \overrightarrow{DE} + \overrightarrow{DG} + \overrightarrow{DF}$ are shown in Figures 8 (a) and (b) respectively below:

$$u := \begin{bmatrix} -0.933319684264525 \\ 0.253884193053021 \\ 0.253884193053021 \end{bmatrix}$$

$$w := \begin{bmatrix} -0.444597151810896 \\ -0.111198397610115 \\ 0.888801602704778 \end{bmatrix}$$

$$F := \begin{bmatrix} -0.538596816523828 \\ 0.350527657832105 \\ 0.350527657832105 \end{bmatrix}$$

$$G := \begin{bmatrix} 0.540260949483891 \\ -0.0524579301479159 \\ 0.419292844070639 \end{bmatrix}$$

$$H := \begin{bmatrix} -1.20281566033994 \\ 0.695513731184189 \\ 1.16726450540274 \end{bmatrix}$$

Figure 8(a). the vectors $\overrightarrow{u}, \overrightarrow{v}$ and the points F, G and H calculated with Maple.

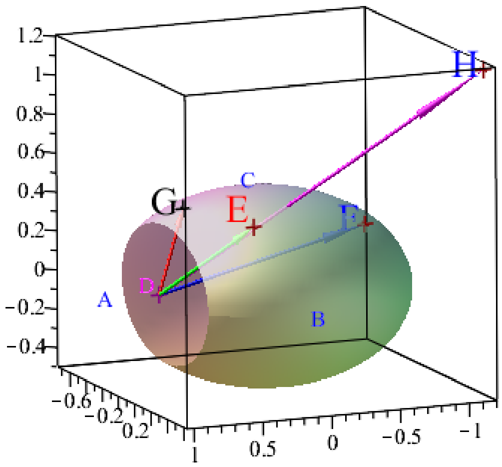


Figure 8(b). The vector of $\overrightarrow{DH} = \overrightarrow{DE} + \overrightarrow{DG} + \overrightarrow{DF}$.

We demonstrate how E is moving along a space curve of $[\sin\varphi, r\cos\varphi\tan\gamma, r\cos\varphi]$ for $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and the angle $\gamma = 45°, \alpha = 30°$ and $\beta = 45°$ in Maple worksheet [S4]. Finally, the locus surface of $\overrightarrow{DE} + \overrightarrow{DG} + \overrightarrow{DF}$ is shown below in Figure 9.

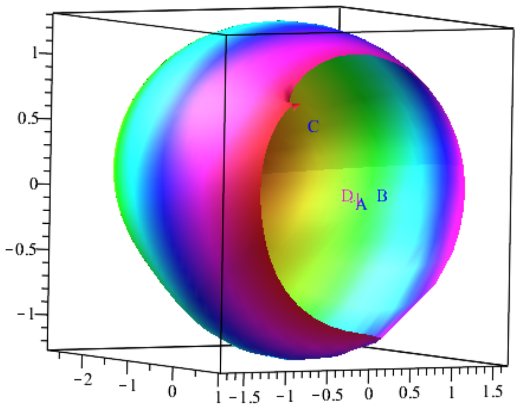


Figure 9. the locus surface of
$\overrightarrow{DE} + \overrightarrow{DG} + \overrightarrow{DF}$.

3.2 Construction of the 3D Locus with the DGS

We see from Section 3.1 that constructing the locus H analytically using the CAS Maple *[3]* requires being able to construct rotation matrices. We shall see in this section how to construct the locus using GInMA *[2]*. We shall see that using a 3D DGS can make a complex 3D concept much more accessible through appreciated and needed 3D visualization, and leave tedious algebraic computations behind, which will make students concentrate on the key concepts of the problem first. We briefly describe a process using GInMA *[2]* to complete this construction.

1. The programing code is organized into blocks. Each code block belongs to one object.
2. There are two types of variables:
   (a) Common variables are for all blocks: These variables use the name of an object or scale. For example, $A.x$ is $x-$coordinate of the point A.
   (b) Non-common variables: These are declared in a code block. These variables are used only in this block and are visible only in the specific code block.
3. Each code block belongs to an object that may depend on other object property (for example, point coordinate).
4. All statements are executed in order, as defined by the order of final construction of the objects. For example, after we make the object A, we can make the object B which depends from A. If C does not depend on A, we can make A depend on C. In this case C will be calculated before A because final A is made after C.
5. Finally, each time a draft GInMA worksheet is drawn or object property is shown, visible results are executed in an order with the following property: For each code block, calculating properties of objects that are associated with this code block will be executed before execution of the whole code block.

We shall use GInMA *[2]* to explore Example 3 below.

Example 3 *We consider the ellipsoid $OABC$ of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Without loss of generality, we assume $a=1$ and we write the ellipsoid $OABC$ with $OA=a=1, 0<OB=b\leq 1, 0<OC=c\leq 1)$ as follows: $\{x,y,z\}=O+|A-O|\cdot\{\sin(\varphi), b\cos(\varphi)\cos(\gamma), c\sin(\varphi\cos(\gamma)\}$ for $\varphi\in[-\pi/2,\pi/2]$ and $\gamma\in[0,2\pi]$, where we refer the angles φ and γ to Figure 3.*

1. We write $D=O+\{d,0,0\}\cdot|A-O|$.We pick $a=1, b=0.5, c=0.5, d=0.84, \alpha=\frac{83\pi}{180}=83^\circ, \beta=\frac{33\pi}{180}=33^\circ$, $\varphi=0.3$ radian and $\gamma=\frac{\pi}{3}=60^\circ$ for demonstration purpose. We refer readers to [S5] for exploration. We use the substitution of $r=\frac{bc}{\sqrt{b^2+c^2\tan^2\gamma}}$, and pick E to be on the space curve of $[\sin\varphi, r\cos\varphi\tan\gamma, r\cos\varphi]$, for $\varphi\in[-\pi,\pi]$ and $\gamma=\frac{\pi}{3}=60^\circ$. For computation purpose using GInMA *[2]*, we convert the angles from degree to radian measurement. Figure 4 shows the point E, the plane OAC and the plane OAE.

2. We choose $\overrightarrow{v_1}=\frac{\overrightarrow{OD}\times.\overrightarrow{DE}}{\|\overrightarrow{OD}\times.\overrightarrow{DE}\|}$ as in Section 3.1, which is the normal vector of the plane OAE. In GInMA *[2]*, we write $\overrightarrow{v_1}=[pe-D, O-D]$, where pe denotes the point E, and we normalize the vector v_1. Similarly, we need $\overrightarrow{v_2}=\frac{\overrightarrow{DE}\times.v_1}{\|\overrightarrow{DE}\times.v_1\|}$. In *[2]*, we write $\overrightarrow{v_2}=[pe-D,\overrightarrow{v_1}]$ and normalize $\overrightarrow{v_2}$.

3. Following the construction leading to Eq. (13) we know the unit vector corresponding to a rotation of $\overrightarrow{DE}$ about the axis $\overrightarrow{v_1}$ by an angle $\alpha=\measuredangle EDF$ is $\overrightarrow{u}=\frac{A(\overrightarrow{DE})}{\|A(\overrightarrow{DE})\|}$. In GInMA *[2]*, we write $\overrightarrow{u}$ as $(rotate(pe, D, v_1, -\alpha\cdot pi/180)-D)$, and normalize the vector $\overrightarrow{u}$. As we recall the purpose of $\overrightarrow{u}$ is to construct the point F that lies on the ellipsoid and the plane OAE with $\measuredangle EDF=\alpha$. Figure 5 shows the vector $\overrightarrow{DF}$ in green, the vector $\overrightarrow{v_1}$ in blue and the vector $\overrightarrow{v_2}$ in magenta.

4. As we have seen in Eq. (14) that the scalars a_{11}, b_{11} and l_1 are needed for expressing the vector $\overrightarrow{OF}$. In GInMA *[2]*, we write $a1=u.x\hat{}2+u.y\hat{}2/b\hat{}2+u.z\hat{}2/c\hat{}2$, $b1=d\cdot u.x/a1$ and $l=sqrt(b1\hat{}2+(1-d\hat{}2)/a1)-b1$. We write the point F as $pf=D+l\cdot u\cdot|A-O|$, which is the Eq. (15) in *[2]*.

5. Similar to how we construct the vector $\overrightarrow{w}$ in Eq. (17), write, in GInMA *[2]*, $\overrightarrow{w}=(rotate(pe, D, v_2, -\beta\cdot pi/180)-D)$ and normalize the vector $\overrightarrow{w}$. This is a rotation of $\overrightarrow{DE}$ around rotation axis v_2 by a pre-determined angle β.

6. To construct, in GInMA *[2]*, the vector $\overrightarrow{OG}$ as shown in Eq. (18), write $a2=v.x\hat{}2+v.y\hat{}2/b\hat{}2+v.z\hat{}2/c\hat{}2$, $b2=d\cdot v.x/a2$ and $l2=sqrt(b2\hat{}2+(1-d\hat{}2)/a2)-b2$, and write the point G as $pg=D+l2\cdot v\cdot|A-O|$.We recall that the point G lies on the ellipsoid and the plane EDG whose normal vector is v_2 and $\measuredangle EDG=\beta$. Figure 6 shows the point G, the vector $\overrightarrow{v_1}$ in blue and the vector $\overrightarrow{v_2}$ in magenta.

7. The locus H is written in GInMA *[2]* as $\{x,y,z\}=pe+pf+pg-2\cdot D$. In other words, $H=E+F+G-2D$. Figures 10(a) and 10(b), respectively, show the space

curve corresponding to this specific locus H and the GInMA *[2]* code that implements the construction.

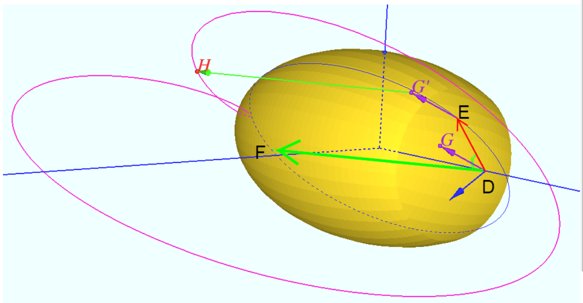
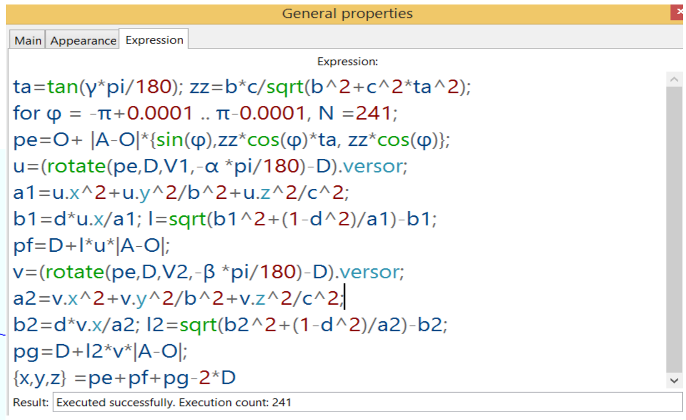


Figure 10(a). Locus H on a space curve.




```
ta=tan(γ*pi/180); zz=b*c/sqrt(b^2+c^2*ta^2);
for φ = -π+0.0001 .. π-0.0001, N =241;
pe=O+ |A-O|*{sin(φ),zz*cos(φ)*ta, zz*cos(φ)};
u=(rotate(pe,D,V1,-α *pi/180)-D).versor;
a1=u.x^2+u.y^2/b^2+u.z^2/c^2;
b1=d*u.x/a1; l=sqrt(b1^2+(1-d^2)/a1)-b1;
pf=D+l*u*|A-O|;
v=(rotate(pe,D,V2,-β *pi/180)-D).versor;
a2=v.x^2+v.y^2/b^2+v.z^2/c^2;
b2=d*v.x/a2; l2=sqrt(b2^2+(1-d^2)/a2)-b2;
pg=D+l2*v*|A-O|;
{x,y,z} =pe+pf+pg-2*D
```

Figure 10(b). The GInMA code for the space curve H.

8. Figures 11(a) and 11(b), respectively, show the locus surface and the GInMA *[2]* code that implements the construction.

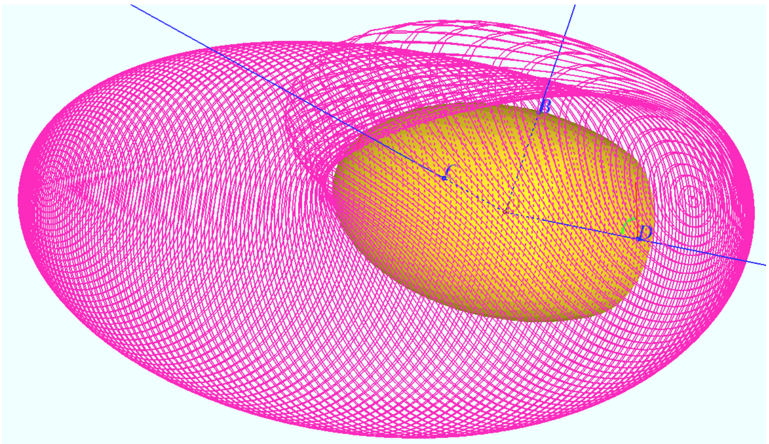


Figure 11(a). Locus surface for H.

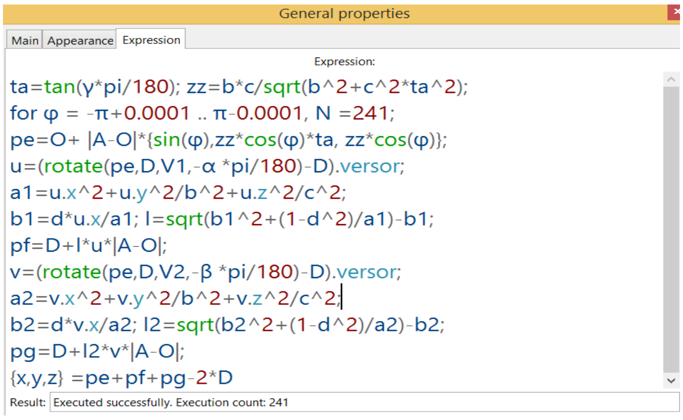


Figure 11(b). The GInMA code for the space curve H.

4 Conclusions

A Google search on the topic of '3D locus' does not produce many hits. Based on this lack of evidence, the authors conjecture that finding solutions to the 3D locus problems may not yet be completely well understood and that these types of problems can be valuable for future exploration or research projects. Therefore, we think that the locus practice problems that appeared at a Chinese college entrance practice exams such as the ones from *[5]* can be an inspiring resource for students and researchers. In this paper, we have seen how a static 2D problem can be extended to other 2D scenarios when a DGS is available for students to explore and a CAS is available for students to verify their conjectures. Authors are fortunate to have a 3D DGS such as GInMA *[2]* for making conjectures about the appearance of a 3D locus. Furthermore, we have seen in Section 3.1 that the complex and tedious computations cannot be realized without the help Maple *[3]*. We recall that we described how to construct a linearly independent set stemming from a fixed D in an ellipsoid systematically but not randomly. It is known that the rotation of vectors or matrices is used frequently in computer graphics, we see how the importance of these concepts are implemented in Section 3. The techniques adopted here can be implemented when investigating various locus problems with other closed surfaces. With a 3D DGS such as GInMA*[2]*, it allows us to drag and rotate, and appreciate the validation from visualization. Since 3D visualization is vital for students, teachers and even researchers for their respective tasks. Further developments in 3D DGS are definitely needed and beneficial to all mathematics communities.

5 Acknowledgements

Authors would like thank reviewers for making several improvements on the paper.

6 Supplementary Electronic Materials

[S1] Maple worksheet for Example 1:
https://mathandtech.org/eJMT_Oct_2017/eJMT_Example1.mws

[S2] GInMA file for Example 1:
https://mathandtech.org/eJMT_Oct_2017/eJMT_Example1.ginma

[S3] Maple worksheet for Example 2:
https://mathandtech.org/eJMT_Oct_2017/eJMT_Example2.mws

[S4] Maple worksheet for animating the point E in Example 2:
https://mathandtech.org/eJMT_Oct_2017/eJMT_Example2_animation.mws

[S5] GInMA file for Example 3:
https://mathandtech.org/eJMT_Oct_2017/eJMT_Example3.ginma

[S6] A video clip for Examples 1 and 3 using GInMA:
https://mathandtech.org/eJMT_Oct_2017/eJMT_Oct_2017.mp4

References

[1] Eberly, D., Computing Orthonormal Sets in 2D, 3D, and 4D, https://www.geometrictools.com/Documentation/OrthonormalSets.pdf.

[2] Geometry in Mathematical Arts (GInMA): *A Dynamic Geometry System*, see http://deoma-cmd.ru/en/Products/Geometry/GInMA.aspx. Free version can be downloaded to experiment GInMA files.

[3] Maple: A product of Maplesoft, see http://maplesoft.com/.

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this URL: http://atcm.mathandtech.org/EP2016/invited/4052016_21300.pdf, ISBN:978-0-9821164-9-4 (hard copy), ISSN 1940-4204 (online version), Mathematics and Technology LLC.Electronic Supplementary Materials